Confidence Limits in Poisson Statistics of Small Numbers
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Richard Selesnick, April 18, 1996

In a previous report (SRL internal report #102) I showed using Bayesian statistics that in a counting experiment sampling from a Poisson distribution the probability density function for the mean number of counts p is just the Poisson distribution evaluated at the observed number n:

$$f(p,n) = \frac{e^{-p}p^n}{n!} \tag{1}$$

Interpreted as a function of p, this is a gamma distribution[1] with parameters $\alpha = n+1$ and $\beta = 1$. In fact, it is well known that the gamma distribution is a conjugate prior for the Poisson distribution (e.g. DeGroot, Probability and Statistics, Addison-Wesley, 1975, p. 268), so that if the experiment is repeated and m counts are measured then the combined probability density function for p is given by a gamma distribution with parameters $\alpha = n+m+1$ and $\beta = 2$ or $f(2p, n+m)$.

Confidence limits on p can be derived from (1). For example, in the previous report I showed that the 68% central confidence interval is, to a good approximation,

$$p = n + 1 \pm \left(n + \frac{3}{4}\right)^{\frac{1}{2}} \tag{2}$$

The central confidence interval is defined such that the probability of p being higher than the upper limit is equal to the probability of p being lower than the lower limit. That is, the interval given by (2) contains 68% of the probability and there is 16% of the probability both above and below it. Because of the asymmetry of the distribution (1), the probability f at the lower limit is higher than the probability f at the upper limit, and there are points within the confidence interval that are less likely than points outside the confidence interval.

Another type of confidence interval that can be defined is one for which the probability f at the lower and upper limits are equal. Then every point

[1] The gamma distribution with parameters α and β is defined as $g(p) = \frac{\beta^\alpha}{\Gamma(\alpha)} p^{\alpha-1} e^{-\beta p}$.

within the confidence interval is more likely than every point outside the confidence interval, but the probability of being above the upper limit is not equal to the probability of being below the lower limit. Also, this definition gives the shortest possible confidence interval for a given probability, so I call it the minimum confidence interval.

As an example, the distribution f with $n = 1$ is shown in Figure 1 along with the central and minimum 68% confidence intervals. A comparison of the two types of confidence intervals in shown in Figure 2, where the lower error bars always correspond to the minimum confidence interval. The data point is plotted at the maximum likelihood point $p = n$. A table of the error bars (the differences between n and the upper and lower confidence limits) is also attached. The minimum confidence intervals were calculated numerically using the Numerical Recipes root finder MNEWT with (2) as a starting point.

The choice of which type of confidence interval to use depends on the application. For quoting an experimental result I prefer the central confidence interval because it is more commonly used and is symmetric about the mean $(n+1)$ of the distribution. For comparison with a model fit (which should be done using the Poisson likelihood χ^2 function, see internal report #102) the minimum confidence interval is best because χ^2 is calculated from point rather than interval probabilities.

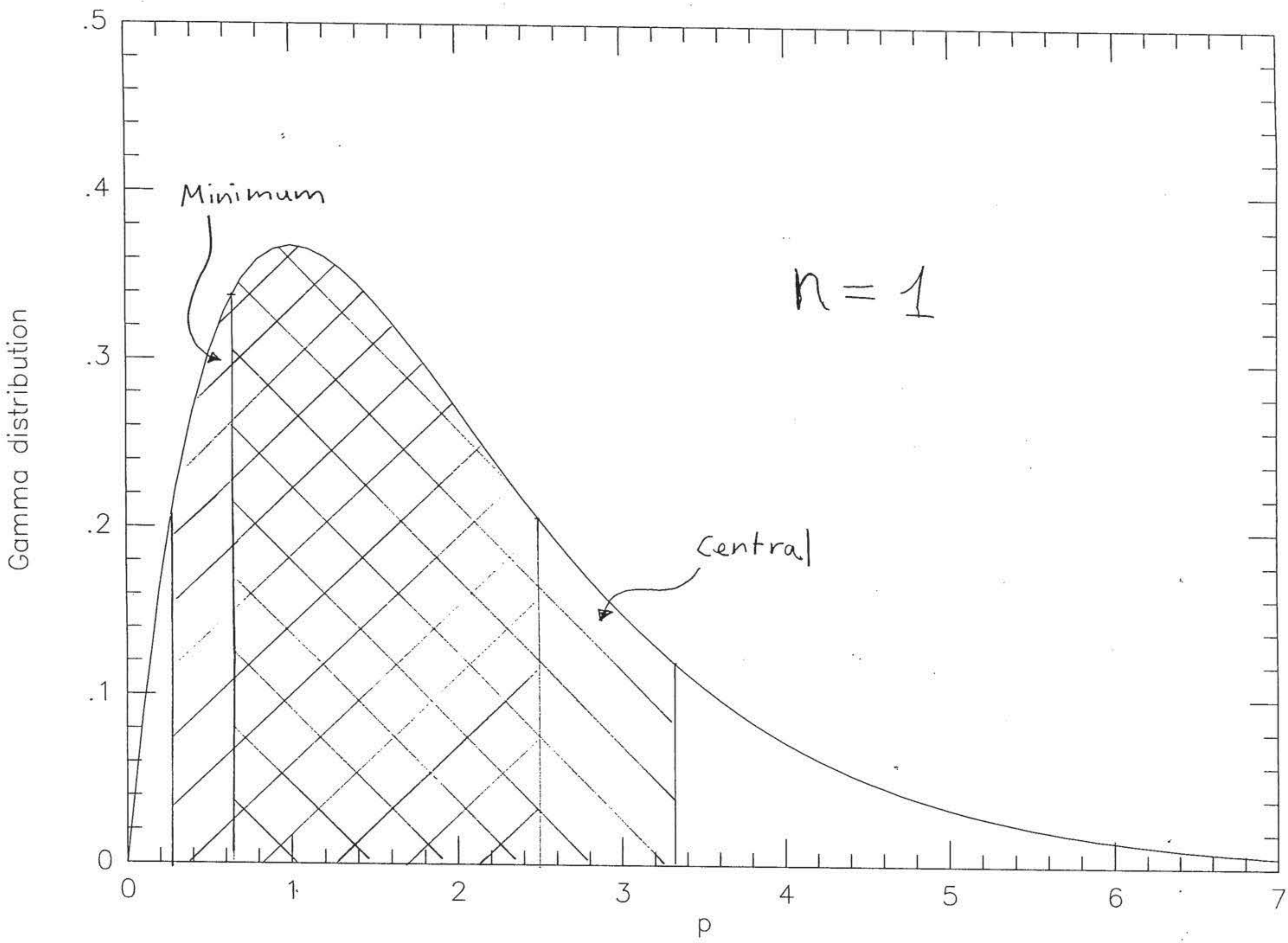

Fig 1

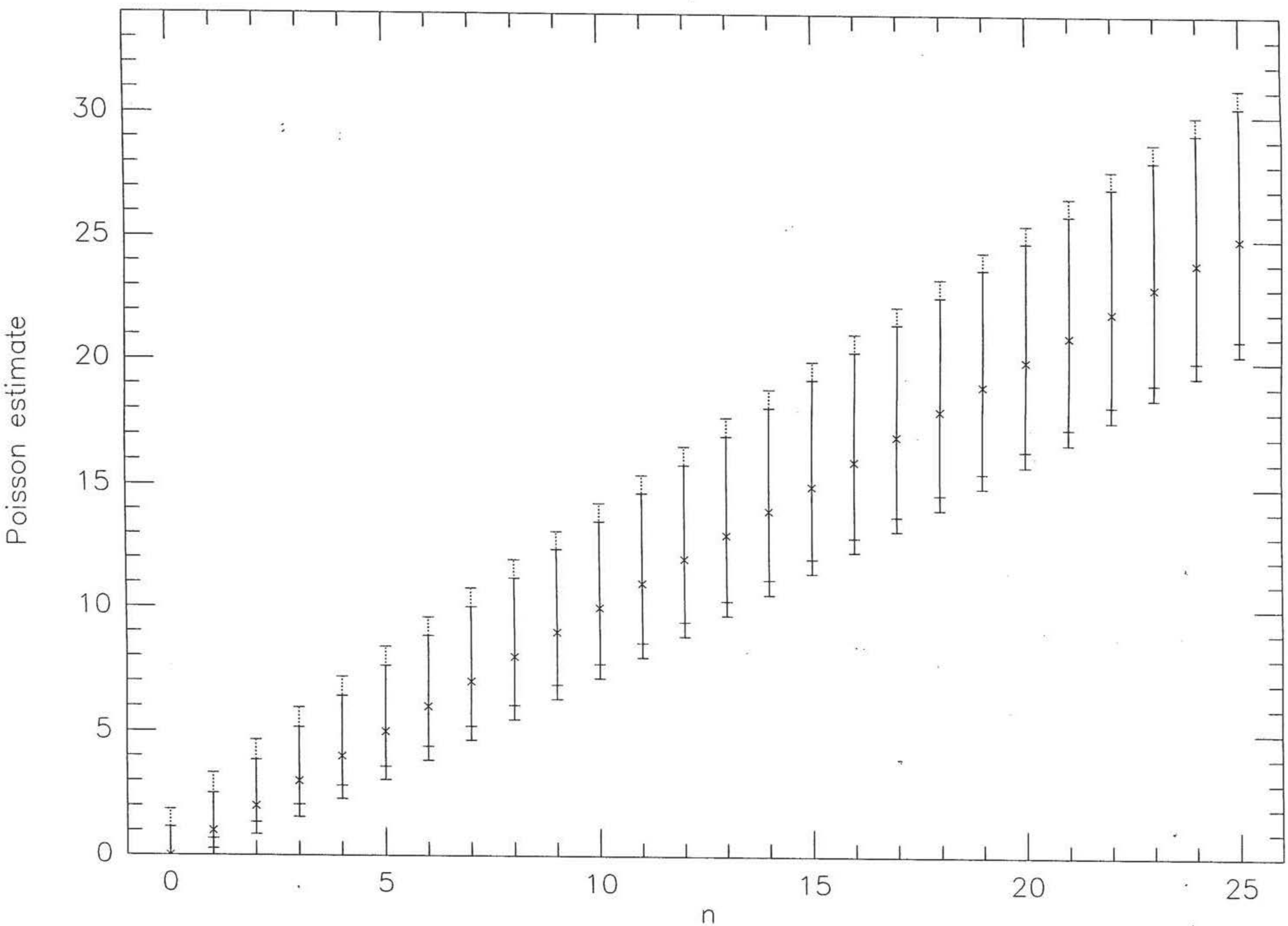

Fig 2

Poisson confidence intervals
(error bars relative to maximum likelihood point n
i.e. upper limit = n + sig+, lower limit = n - sig-)

	approx central		minimum	
n	sig-	sig+	sig-	sig+
0	0.	1.86603	0.	1.14788
1	0.322876	2.32288	0.731756	1.50066
2	0.658312	2.65831	1.13574	1.85450
3	0.936492	2.93649	1.44686	2.14831
4	1.17945	3.17945	1.71012	2.40286
5	1.39792	3.39792	1.94269	2.63020
6	1.59808	3.59808	2.15337	2.83740
7	1.78388	3.78388	2.34740	3.02894
8	1.95804	3.95804	2.52822	3.20790
9	2.12250	4.12250	2.69822	3.37645
10	2.27872	4.27872	2.85913	3.53620
11	2.42783	4.42783	3.01229	3.68841
12	2.57071	4.57071	3.15870	3.83404
13	2.70810	4.70810	3.29921	3.97387
14	2.84057	4.84057	3.43446	4.10856
15	2.96863	4.96863	3.56502	4.23862
16	3.09268	5.09268	3.69133	4.36450
17	3.21307	5.21307	3.81378	4.48656
18	3.33013	5.33013	3.93272	4.60516
19	3.44410	5.44410	4.04842	4.72055
20	3.55522	5.55522	4.16115	4.83300
21	3.66369	5.66369	4.27111	4.94272
22	3.76970	5.76970	4.37849	5.04988
23	3.87340	5.87340	4.48349	5.15468
24	3.97494	5.97494	4.58623	5.25722
25	4.07445	6.07445	4.68689	5.35772