REVISED - September, 1979

Internal Report No. 4

Error Bars for Poisson Statistics of Small Numbers

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September 1968

Summary:

For a gaussian distribution, error bars plotted at $\pm\sigma$ give the 84.13 percent confidence, upper and lower limits. For a Poisson distributed quantity, such as the number of cosmic rays incident on a detector in a given time, a very good estimate for the 84.13 percent confidence, upper and lower limits for N detected events is:

$$\text{lower limit:} \quad N - \sqrt{N - 0.25} \qquad n \geq 1$$

$$\text{upper limit:} \quad N + \sqrt{N + 0.75} + 1 \qquad n \geq 0$$

i.e, the length of the lower error bar is $\sqrt{N - 0.25}$ and the length of the upper bar is $\sqrt{N + 0.75} + 1$. Note that as $N \to \infty$ the error bars $\to \pm\sqrt{N}$. The following table indicates the accuracy of this approximation:

Length of Error Bar to 84.13 Percent Confidence Point

N	$\sqrt{N}$	lower error bar		upper error bar	
		exact	$\sqrt{N - 0.25}$	exact	$\sqrt{N + 0.75} + 1$
0	0	-	-	1.841	1.865
1	1.00	0.827	0.865	2.30	2.32
2	1.41	1.30	1.32	2.64	2.66
3	1.73	1.64	1.66		
.					
.					
.					
9	3.00	2.94	2.96	4.13	4.12
.					
.					
.					
15	3.88			4.96	4.96
16	4.00	3.96	3.96		

<u>Derivation:</u> (Page numbers for reference refer to CRC Handbook of Probability & Statistics.)

For Poisson statistics, if you observe x' total counts, the lower limit, m_a, with α significance (i.e. 1 - α confidence level) is given by (page 190)

$$\sum_{x = x'}^{\infty} \frac{e^{-m_a} m_a^{x}}{x!} = \alpha \qquad (1)$$

and the corresponding upper limit, m_b, is given by

$$\sum_{x = 0}^{x'} \frac{e^{-m_b} m_b^{x}}{x!} = \alpha \qquad (2)$$

One can find any desired confidence level exactly using eq. 1 or 2 and the table of Cumulative Terms of the Poisson Distribution (page 181). The confidence levels can also be found exactly from a table of χ^2 probability using the relation (page 231)

$$1 - P(\chi^2;n) = \sum_{x = 0}^{x' - 1} \frac{e^{-m} m^{x}}{x!} \qquad (3)$$

where $P(\chi^2;n)$ is the χ^2 probability for n degrees of freedom with χ^2 = 2m and n = 2x'. Then m_a and m_b are given by

$$\alpha = P\ (2m_a;2x') \qquad (4)$$

$$1 - \alpha = P\ (2m_b;2(x' + 1)) \qquad (5)$$

To find an approximate formula for m_a and m_b we note that (page 231)

$$\chi^2_\alpha = \frac{1}{2}(x_\alpha + \sqrt{2n - 1})^2 \qquad (6)$$

for $n > 30$ where χ^2_α is the value of χ^2 such that the χ^2 probability is α, (with n degrees of freedom) and x_α is the value of x such that the normal distribution $\phi(x)$ has the same probability α. (In other words $\sqrt{2\chi^2} - \sqrt{2n - 1}$ is a normal deviate with unit variance.) We are interested in the value of m_a corresponding to 1 σ in a gaussian distribution so we set $x_\alpha = 1$ in eq. 6. Then with $\chi^2_\alpha = 2\ m_a$ and $n = 2\ x'$ we find

$$m_a = x' - \sqrt{x' - 0.25}\ .$$

Similarly with $\chi^2_\alpha = 2\ m_b$ and $n = 2(x' + 1)$ we find

$$m_b = x' + \sqrt{x' + 0.75} + 1\ .$$

These are the formulae given in the summary (with x' replacing N). Note that these formulae were derived for $n > 30$, i.e. $N > 15$, but the table demonstrates that they give a good approximation even for $N = 0$ or 1.

Note that by substituting $x_\alpha = s$ into eq. 6 we can find the upper and lower limits corresponding to s σ in a gaussian distribution. Let $\ell_a = \sqrt{N - 0.25}$ and $\ell_b = \sqrt{N + 0.75} + 1$. Then the length of the error bar corresponding to s standard deviations in a gaussian distribution is:

$$\text{lower bar:} \quad s\ell_a - \frac{s^2 - 1}{4}$$

$$\text{upper bar:} \quad s\ell_b + \frac{(s - 1)(s - 3)}{4}$$

In particular the "2 σ" error bars in this approximation are $2\ell_a - 0.75$ and $2\ell_b - 0.25$, where ℓ_a and ℓ_b are the "1 σ" error bars.

Supplement to Internal Report #4

Error Bars for Poisson Statistics by Newton's Method

Bruce Prickett July 1979

While the aproximations derived in Report #4 are acurate to 1%, it is possible to find the exact values of the error bars to an arbitrary precision using Newton's method in a fairly simple computer program. By differentiating eqs. #1 and #2, we find

$$\frac{da}{dma} = \frac{ma^{x-1}\, e^{-ma}}{(x-1)!} \qquad \frac{da}{dmb} = \frac{-e^{-mb}\, mb^{x}}{x!}$$

To get the values of m for sigma=1,2,3, we set a equal to .1587, .023, and .00135. We then aproximate ma and mb as $\sqrt{x-.25}$ and $\sqrt{x+.75}$, and find the sums given by eqs. 1 and 2 to an arbitrary precision. These sums are then subtracted from a, and the difference multiplied by dm/da to find the change in m. By succesive iterations, any degree of precision can be achieved.

x	$\sqrt{x}$	lower error bars $x-\sqrt{x-.25}$	s=1	s=2	s=3	upper error bars $\sqrt{x+.75}-x$	s=1	s=2	s=3
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	1.8660	1.8407	3.7723	6.6077
1.0	1.0000	0.8660	0.8272	0.9767	0.9986	2.3229	2.2992	4.6699	7.9002
2.0	1.4142	1.3229	1.2917	1.7685	1.9471	2.6583	2.6374	5.3341	8.8695
3.0	1.7321	1.6583	1.6325	2.4012	2.7883	2.9365	2.9177	5.8861	9.6805
4.0	2.0000	1.9365	1.9141	2.9384	3.5347	3.1794	3.1622	6.3688	10.3924
5.0	2.2361	2.1794	2.1594	3.4125	4.2081	3.3979	3.3819	6.8030	11.0348
6.0	2.4495	2.3979	2.3796	3.8412	4.8250	3.5981	3.5830	7.2011	11.6248
7.0	2.6458	2.5981	2.5811	4.2354	5.3970	3.7839	3.7696	7.5707	12.1736
8.0	2.8284	2.7839	2.7680	4.6021	5.9323	3.9580	3.9445	7.9172	12.6887
9.0	3.0000	2.9580	2.9430	4.9465	6.4370	4.1225	4.1095	8.2446	13.1758
10.0	3.1623	3.1225	3.1082	5.2720	6.9157	4.2787	4.2662	8.5556	13.6389
11.0	3.3166	3.2787	3.2651	5.5816	7.3721	4.4278	4.4158	8.8524	14.0814
12.0	3.4641	3.4278	3.4147	5.8773	7.8089	4.5707	4.5590	9.1370	14.5057
13.0	3.6056	3.5707	3.5581	6.1608	8.2285	4.7081	4.6968	9.4106	14.9139
14.0	3.7417	3.7081	3.6959	6.4336	8.6326	4.8406	4.8295	9.6744	15.3077
15.0	3.8730	3.8406	3.8288	6.6967	9.0229	4.9686	4.9579	9.9295	15.6886
16.0	4.0000	3.9686	3.9572	6.9512	9.4008	5.0927	5.0822	10.1766	16.0578
17.0	4.1231	4.0927	4.0815	7.1977	9.7672	5.2131	5.2028	10.4165	16.4162
18.0	4.2426	4.2131	4.2022	7.4371	10.1233	5.3301	5.3201	10.6497	16.7648
19.0	4.3589	4.3301	4.3195	7.6699	10.4697	5.4441	5.4343	10.8768	17.1043
20.0	4.4721	4.4441	4.4337	7.8966	10.8073	5.5552	5.5456	11.0982	17.4355
21.0	4.5826	4.5552	4.5451	8.1176	11.1367	5.6637	5.6542	11.3144	17.7588
22.0	4.6904	4.6637	4.6537	8.3334	11.4584	5.7697	5.7604	11.5257	18.0748
23.0	4.7958	4.7697	4.7599	8.5444	11.7731	5.8734	5.8642	11.7323	18.3841
24.0	4.8990	4.8734	4.8638	8.7508	12.0810	5.9749	5.9659	11.9347	18.6870
25.0	5.0000	4.9749	4.9655	8.9529	12.3826	6.0744	6.0655	12.1330	18.9839
26.0	5.0990	5.0744	5.0652	9.1510	12.6784	6.1720	6.1632	12.3276	19.2751
27.0	5.1962	5.1720	5.1629	9.3453	12.9686	6.2678	6.2591	12.5185	19.5610
28.0	5.2915	5.2678	5.2588	9.5360	13.2535	6.3619	6.3533	12.7060	19.8418