Geometrical Factors

Space Radiation Lab Internal
Report Number 7

Tom Garrard

Solid Angle between Angles

$\Omega(\theta_1, \theta_2) \equiv$ solid angle between polar angles θ_1 and θ_2.

$$= \int_{\theta_1}^{\theta_2} 2\pi \sin\theta \, d\theta$$

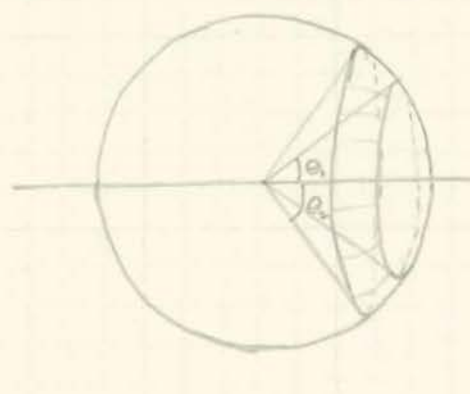
$$= 2\pi \left| \cos\theta_1 - \cos\theta_2 \right|$$

$$= 4\pi \left| \sin(\tfrac{1}{2}(\theta_1+\theta_2)) \sin(\tfrac{1}{2}(\theta_1-\theta_2)) \right|$$

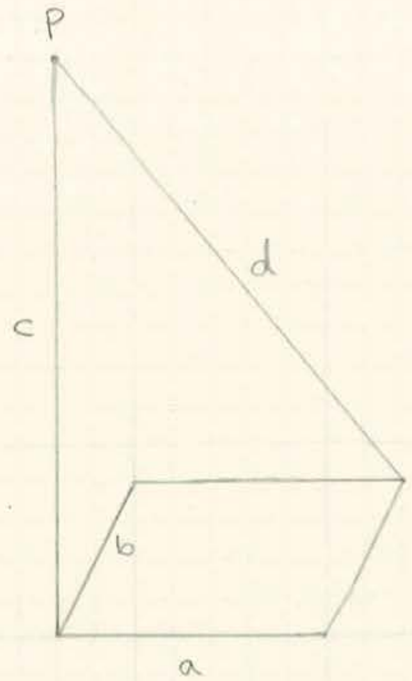
$\Omega(0°, 10°) = .03\pi = .75\%$ of sphere

$\Omega(10°, 20°) = .09\pi = 2.25\%$ of sphere

$\vdots$



Solid Angle of Rectangle



line c is $\perp$ to plane ab

Ω = solid angle subtended by area ab at point P

$$\Omega = \int_{ab} \frac{dA \cos\theta}{r^2}$$

$\alpha = a/c$ $\qquad\qquad$ $\cos\theta = c/r$

$\beta = b/c$

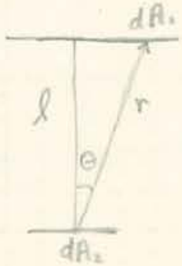
$$\Omega = \int_0^\alpha dx \int_0^\beta dY \frac{1}{(1+X^2+Y^2)^{3/2}}$$

$$= \tan^{-1} \frac{\alpha\beta}{\sqrt{1+\alpha^2+\beta^2}}$$

$$= \tan^{-1} \frac{ab}{cd} = \tan^{-1} \frac{ab}{c\sqrt{a^2+b^2+c^2}}$$

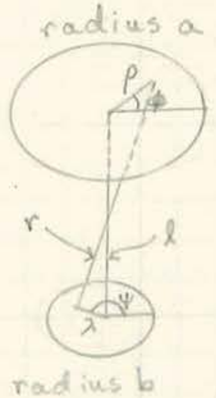
This can be checked by expanding the integrand above with the binomial theorem, integrating, and summing the series. See also UCRL 2426 (1966 ed.).

G. F. of Two Discs



$$S = \iint d\Omega \, dA$$

$$= \iint dA_1 \frac{\cos\theta}{r^2} dA_2 \cos\theta$$



$$A = a/\ell \qquad \alpha = \rho/\ell$$
$$B = b/\ell \qquad \beta = \lambda/\ell$$

$$r^2 = \ell^2 \left[1 + (\alpha\cos\phi - \beta\cos\psi)^2 + (\alpha\sin\phi - \beta\sin\psi)^2 \right]$$
$$= \ell^2 Q^2$$

$$\cos\theta = \frac{\ell}{r}$$

$$dA_1 = \ell^2 \, \alpha \, d\alpha \, d\phi$$
$$dA_2 = \ell^2 \, \beta \, d\beta \, d\psi$$

$$S = \ell^2 \iint \frac{\alpha \, d\alpha \, d\phi \, \beta \, d\beta \, d\psi}{Q^4}$$

$$Q^2 = 1 + \alpha^2\cos^2\phi + \beta^2\cos^2\psi - 2\alpha\beta\cos\phi\cos\psi + \alpha^2\sin^2\phi + \beta^2\sin^2\psi - 2\alpha\beta\sin\phi\sin\psi$$

$$= 1 + \alpha^2 + \beta^2 - 2\alpha\beta\cos(\phi-\psi)$$

$$= \mu + \nu\cos(\phi-\psi)$$

$$\frac{S}{\ell^2} = \int_0^A \alpha \, d\alpha \int_0^B \beta \, d\beta \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \, \frac{1}{(\mu + \nu\cos(\phi-\psi))^2}$$

$$= \int_0^A \alpha \, d\alpha \int_0^B \beta \, d\beta \int_0^{2\pi} d\psi \; I_1$$

It should be clear from the drawing that I_1 is independent of ψ. We therefore evaluate I_1 for a convenient value of ψ and multiply by 2π to get the integral over ψ. Thus, let $\psi = 0$ and

$$I_1 = \int_0^{2\pi} \frac{d\phi}{(\mu + \nu\cos\phi)^2}$$

$$= 2\int_0^{\pi} \frac{d\phi}{(\mu + \nu\cos\phi)^2}$$

$$= 2\left\{ \frac{\nu \sin\phi}{(\nu^2-\mu^2)(\mu+\nu\cos\phi)} - \frac{2\mu}{(\mu^2-\nu^2)^{3/2}} \tan^{-1}\frac{(\mu-\nu)\tan\phi/2}{(\mu^2-\nu^2)^{1/2}} \right\}\Bigg|_0^{\pi}$$

$$= \frac{2\mu\pi}{(\mu^2-\nu^2)^{3/2}}$$

(Use Dwight 446.03 + 446.00, noting that $\mu^2 > \nu^2$.)

$$\frac{S}{\ell^2} = \int_0^A \alpha\, d\alpha \int_0^B \beta\, d\beta \; \frac{4\pi^2\mu}{(\mu^2-\nu^2)^{3/2}}$$

$$= \int_0^A \alpha\, d\alpha \; I_2 \, (4\pi^2)$$

$$I_2 = \int_0^B d\beta \; \frac{\beta\,(1+\alpha^2+\beta^2)}{((1+\alpha^2+\beta^2)^2 - 4\alpha^2\beta^2)^{3/2}}$$

$$1+\alpha^2 = Z^2$$

$$\beta^2 + Z^2 = X \qquad\qquad \beta = B \Rightarrow X = B^2+Z^2$$
$$\beta\, d\beta = \frac{dX}{2} \qquad\qquad \beta = 0 \Rightarrow X = Z^2$$

$$I_2 = \frac{1}{2}\int_{Z^2}^{B^2+Z^2} \frac{X\,dX}{\left[X^2 - 4\alpha^2 X + 4\alpha^2 Z^2\right]^{3/2}}$$

$$= \left\{ \frac{4\alpha^2 X - 8\alpha^2 Z^2}{(16\alpha^2 Z^2 - 16\alpha^4)(X^2 - 4\alpha^2 X + 4\alpha^2 Z^2)^{1/2}} \right\} \Bigg|_{Z^2}^{B^2+Z^2}$$

$$= \left\{ \frac{1}{4}\,\frac{X - 2Z^2}{(X^2 - 4\alpha^2 X + 4\alpha^2 Z^2)^{1/2}} \right\} \Bigg|_{Z^2}^{B^2+Z^2} \qquad \text{Dwight } 380.013$$

$$I_2 = \frac{1}{4}\left[\frac{B^2 - Z^2}{((B^2+Z^2)^2 - 4\alpha^2 B^2)^{1/2}} + 1\right]$$

$$S = \pi^2 \ell^2 \int_0^A \alpha\, d\alpha$$
$$- \pi^2 \ell^2 \int_0^A \alpha\, d\alpha \; \frac{(1+\alpha^2-B^2)}{((1+\alpha^2+B^2)^2 - 4\alpha^2 B^2)^{\frac{1}{2}}}$$
$$= \pi^2 \ell^2 \left(\frac{A^2}{2} - I_3\right)$$

$$1 + B^2 = R^2$$

$$R^2 + \alpha^2 = s \qquad\qquad \alpha = 0 \Rightarrow s = R^2$$

$$\alpha\, d\alpha = \frac{ds}{2} \qquad\qquad \alpha = A \Rightarrow s^2 = R^2 + A^2$$

$$I_3 = \frac{1}{2}\int_{R^2}^{R^2+A^2} \frac{ds\,(s-2B^2)}{(s^2-4B^2s+4B^2R^2)^{\frac{1}{2}}}$$

Dwight
380.011

$$I_3 = \frac{1}{2}\left\{(s^2-4B^2s+4B^2R^2)^{\frac{1}{2}}\right\}\Big|_{R^2}^{R^2+A^2}$$

$$= \frac{1}{2}\left[\left((R^2+A^2)^2-4B^2A^2\right)^{\frac{1}{2}} - R^2\right]$$

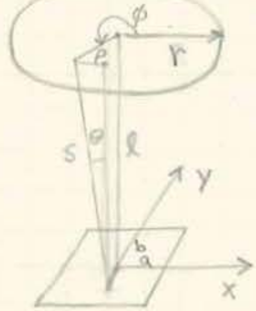
$$= \frac{1}{2}\left[\left(1+B^4+A^4+2B^2+2A^2-2B^2A^2\right)^{\frac{1}{2}} - 1 - B^2\right]$$

$$S = \frac{\pi^2 \ell^2}{2}\left[1+A^2+B^2-\sqrt{(1+A^2+B^2)^2-4B^2A^2}\right]$$

$$S = \frac{\pi^2}{2}\left[\ell^2+a^2+b^2-\sqrt{(\ell^2+a^2+b^2)^2-4a^2b^2}\right]$$

See Newell, Rev. Sci. Instr. 19, 384, (1948) for "Geometrical Factors Underlying Coincidence Counting with Geiger Counters"

G.F. of Disc and Rectangle



$$l = r\cos\theta$$

$$s^2 = l^2 + (\rho\cos\phi - x)^2 + (\rho\sin\phi - y)^2$$

$$S = l^2 \int_{-a}^{a} dx \int_{-b}^{b} dy \int_0^{2\pi} d\phi \int_0^{r} \rho\, d\rho \,\frac{1}{s^4}$$

$\alpha = x/l$ $\qquad A = a/l$ $\qquad \mathsf{s} = \dfrac{s}{l}$

$\beta = y/l$ $\qquad B = b/l$

$\mathcal{R} = \rho/l$ $\qquad R = r/l$

$$S = l^2 \int_{-A}^{A} d\alpha \int_{-B}^{B} d\beta \int_0^{2\pi} d\phi \int_0^{R} \mathcal{R}\, d\mathcal{R} \,\frac{1}{(1 + (\mathcal{R}\cos\phi - \alpha)^2 + (\mathcal{R}\sin\phi - \beta)^2)^2}$$

$$\mathsf{s}^{-4} = (1 + \mathcal{R}^2 + \alpha^2 + \beta^2 - 2\alpha\mathcal{R}\cos\phi - 2\beta\mathcal{R}\sin\phi)^{-2}$$

Assume $\alpha, \beta, R < 1$

$$\begin{aligned}\mathsf{s}^{-4} = 1 &- 2\mathcal{R}^2 - 2\alpha^2 - 2\beta^2 + 4\alpha\mathcal{R}\cos\phi + 4\beta\mathcal{R}\sin\phi \\ &+ 3\mathcal{R}^4 + 3\alpha^4 + 3\beta^4 + 12\alpha^2\mathcal{R}^2\cos^2\phi + 12\beta^2\mathcal{R}^2\sin^2\phi \\ &+ 6\alpha^2\mathcal{R}^2 + 6\beta^2\mathcal{R}^2 + 6\alpha^2\beta^2 - 12\alpha\mathcal{R}^3\cos\phi - 12\beta\mathcal{R}^3\sin\phi \\ &- 12\alpha^3\mathcal{R}\cos\phi - 12\beta^3\mathcal{R}\sin\phi - 12\alpha\mathcal{R}\beta^2\cos\phi - 12\alpha^2\beta\mathcal{R}\sin\phi \\ &+ 12\alpha\beta\mathcal{R}^2\sin\phi\cos\phi \quad + \quad O(6)\end{aligned}$$

Do $\int d\phi$ first.

$$S = \ell^2 \int_{-A}^{A} d\alpha \int_{-B}^{B} d\beta \int_0^R r\, dr \quad F$$

$$F = 2\pi \left\{ 1 - 2r^2 - 2\alpha^2 - 2\beta^2 + 3r^4 + 3\alpha^4 + 3\beta^4 + 6\alpha^2 r^2 \right.$$
$$\left. + 6\beta^2 r^2 + 6\alpha^2 r^2 + 6\beta^2 r^2 + 6\alpha^2\beta^2 \right\} + \cdots$$

$$= 2\pi \left\{ 1 - 2r^2 - 2\alpha^2 - 2\beta^2 + 3r^4 + 3\alpha^4 + 3\beta^4 + 6\alpha^2\beta^2 + \right.$$
$$\left. + 12\alpha^2 r^2 + 12\beta^2 r^2 \right\}$$

Do $\int dr$.

$$S = \ell^2 \int d\alpha \int d\beta \, (2\pi) \, G$$

$$G = \frac{R^2}{2} - \frac{R^4}{2} - \alpha^2 R^2 - \beta^2 R^2 + \frac{R^6}{2} + \frac{3\alpha^4 R^2}{2}$$
$$+ \frac{3\beta^4 R^2}{2} + 3\alpha^2\beta^2 R^2 + 3\alpha^2 R^4 + 3\beta^2 R^4$$

$$S = \ell^2 \, 2\pi \int_{-A}^{A} d\alpha \quad H$$

$$H = R^2 B - R^4 B - 2\alpha^2 R^2 B - \frac{2}{3} R^2 B^3 + R^6 B + 3\alpha^4 R^2 B$$
$$+ \frac{3}{5} B^5 R^2 + 2\alpha^2 R^2 B^3 + 6\alpha^2 R^4 B + 2 B^3 R^4$$

$$S = 4\pi \ell^2 A B R^2 \left\{ 1 - R^2 - \frac{2}{3} A^2 - \frac{2}{3} B^2 + R^4 + \frac{3}{5} A^4 \right.$$
$$\left. + \frac{3}{5} B^4 + \frac{2}{3} A^2 B^2 + 2 A^2 R^2 + 2 B^2 R^2 \right\}$$

The computer programs AOHM, AMOE, and various modifications thereof are available for calculating geometrical factors of stacks of disks or annular detectors. The analytic expression is used for the case of two disks, in other cases, a numerical integration is necessary.

See Write-Up No. 5 for further information on these programs.