ROTATION MATRICES

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Consider 2 coordinate systems:

s: axes x, y, z (For example, HEAO spacecraft coordinates)

G: axes E, N, V (For example, a zenith-azimuth system referenced to earth where E=east, N=north, V=vertical)

Unit vectors represented by $\mathbf{e}_x$, $\mathbf{e}_y$, ...

Vectors represented by column notation, i.e.,

$$\overrightarrow{R_s} = R_x \mathbf{e}_x + R_y \mathbf{e}_y + R_z \mathbf{e}_z = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} \\ \overrightarrow{R} \\ \\ \end{pmatrix}_s$$

The subscript (s, G) on the vector specifies the coordinate system in which the components are given. Coordinate system representations are transformed by an orthonormal rotation matrix, $\overline{\overline{A}}$.

$$\overrightarrow{R_g} = \overline{\overline{A}}\, \overrightarrow{R_s}$$

or

$$\begin{pmatrix} R_E \\ R_N \\ R_V \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}$$

Some relevant properties of rotation matrices are:

$\overline{\overline{A}}^{-1} = \overline{\overline{A^t}}$ (inverse of $\overline{\overline{A}}$ = transpose of $\overline{\overline{A}}$)

$A_{ij}A_{ik} = \delta_{jk}$ (using summation convention for i)

$= A_{ki}A_{ji}$

Note also that

$$\begin{pmatrix} \mathbf{e}_x \end{pmatrix}_G = \overline{\overline{A}} \begin{pmatrix} \mathbf{e}_x \end{pmatrix}_s = \overline{\overline{A}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_s = \begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix}_G$$

Thus we write

$$\overline{\overline{A}} = \begin{pmatrix} \begin{pmatrix} \mathbf{e}_x \end{pmatrix}_G & \begin{pmatrix} \mathbf{e}_y \end{pmatrix}_G & \begin{pmatrix} \mathbf{e}_z \end{pmatrix}_G \end{pmatrix}$$

Similarly $\overline{\overline{A}} = \begin{pmatrix} \overline{\mathbf{e}_E}_s \\ \overline{\mathbf{e}_N}_s \\ \overline{\mathbf{e}_V}_s \end{pmatrix}$

Note that

$$\mathbf{e}_E \cdot \mathbf{e}_x = \mathbf{e}_E \cdot \begin{pmatrix} \mathbf{e}_x \end{pmatrix}_G = A_{11} \text{ , etc.}$$

So that we may write

$$\overline{\overline{A}} = \begin{pmatrix} \mathbf{e}_x \cdot e_E & \mathbf{e}_y \cdot e_E & \mathbf{e}_z \cdot e_E \\ \mathbf{e}_x \cdot e_N & \mathbf{e}_y \cdot e_N & \mathbf{e}_z \cdot e_N \\ \mathbf{e}_x \cdot e_V & \mathbf{e}_y \cdot e_V & \mathbf{e}_z \cdot e_V \end{pmatrix}$$

Since, for example,

$$\mathbb{e}_x = \mathbb{e}_y \times \mathbb{e}_z$$

we can find the third row of any rotation matrix in terms of the other two.

Continuing the example.

$$\begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix} \times \begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \end{pmatrix} \qquad \text{or}$$

$$A_{11} = A_{22}\,A_{33} - A_{32}\,A_{23}$$

$$A_{21} = A_{32}\,A_{13} - A_{12}\,A_{33}$$

$$A_{31} = A_{12}\,A_{23} - A_{22}\,A_{13}$$

Finally note that any rotation matrix may be represented by 3 Euler angles Θ, ϕ, ψ as illustrated on p. 107 of Goldstein's Classical Mechanics.

$$\bar{\bar{A}} = \begin{pmatrix} c\psi c\phi - c\Theta s\phi s\psi & c\psi s\phi + c\Theta c\phi s\psi & s\psi s\Theta \\ -s\psi c\phi - c\Theta s\phi c\psi & -s\psi s\phi + c\Theta c\phi c\psi & c\psi s\Theta \\ s\Theta\, s\phi & -s\Theta\, c\phi & c\Theta \end{pmatrix}$$

where $c\psi$ is shorthand for cosine of ψ, ...

We note that $0^o \leq \Theta \leq 180^o$ (see the figure), and $\Theta = \cos^{-1}(A_{33})$.

Then $\quad \phi = \tan^{-1}\left(A_{31} / (-A_{32})\right)$

and $\quad \psi = \tan^{-1}(A_{13} / A_{23})$

where the quadrants for ϕ and ψ are determined by the signs of the two terms in the argument of the arc tangent (similar to IBM FORTRAN library function ATAN2).