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A Comparison of the Standard and Maximum Likelihood Error Bars for Poisson Statistics

by

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Abstract

The standard and maximum likelihood upper limits of confidence ε for Poisson statistics are the same, but the lower limits differ. It is found that only the standard lower limits satisfy the condition that $\varepsilon\%$ of the observers measuring a fixed rate of events (with measurements Poisson distributed about the rate) have the rate above their lower limits.

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Definitions

The standard definitions of the Poisson upper and lower limits, μ_u and μ_l, of confidence ε for a measured rate of n events per unit time are

$$\sum_{k=0}^{n} \frac{e^{-\mu_u} \mu_u^k}{k!} = 1 - \varepsilon \tag{1a}$$

and

$$\sum_{k=n}^{\infty} \frac{e^{-\mu_l} \mu_l^k}{k!} = 1 - \varepsilon \tag{1b}$$

(CRC Handbook, 1968; Kelly et al., 1980), where $\varepsilon = 0.8413$ for 1 σ error bars. In Internal Report #4 (Israel, 1968) approximate expressions for μ_u and μ_l are obtained from these equations. The maximum likelihood formalism on the other hand leads to the following definitions of μ_u and μ_l:

$$\sum_{k=0}^{n} \frac{e^{-\mu_u} \mu_u^k}{k!} = 1 - \varepsilon \tag{2a}$$

and

$$\sum_{k=n+1}^{\infty} \frac{e^{-\mu_l} \mu_l^k}{k!} = 1 - \varepsilon \tag{2b}$$

(see the Appendix).

Equations (1a) and (2a) show that the upper limits are the same in both cases. The lower limits (equations (1b) and (2b)), however, are different, with the standard lower limit always lower than that obtained with the maximum likelihood method.

The Test

In order for limits of confidence ε to be meaningful, one would like them to be assigned in such a way that, given a large number of measurements of a fixed rate of events, ε% of the measurements (Poisson distributed about the rate) have the rate above (below) their lower (upper) limit. This requirement is, in fact, used to define confidence intervals in Mathematical Methods of Statistics (Cramér, 1974, p.512; careful, his ε is one minus the ε used here). This book also points out that, for a discrete distribution function such as a Poisson distribution, the condition that "ε% of the measurements have the rate above their lower limit" must by replaced by "at least ε% of the measurements . . ."

A computer program has been written to test both the standard and maximum likelihood upper and lower limits to see which ones, if any, satisfy the above requirement. The test was performed as follows:

An average event rate, μ, was chosen and the Poisson distribution of measurements about that rate calculated. For example, for $\mu = 9.5$ events per unit time, the fraction of observers measuring 8 events in a unit time is $e^{-9.5}\, 9.5^8 / 8! = 0.12$. This fraction was calculated for all possible measured values 0, 1, ..., p (where $p \equiv$ a number $\gg \mu$). For each measurement, upper and lower limits were assigned according to either the standard or maximum likelihood definition. The confidence level chosen for the test was $\varepsilon = 84.13\%$. The total fraction of observers (the sum over all possible measured values) who had the average rate above (below) their lower (upper) limit was then determined, and that number was plotted as a function of μ. This calculation was performed for all average rates between 0 and 150 events per unit time in steps of 0.2.

Results

The results for the standard and maximum likelihood upper limits are of course identical since the definitions are the same, and are shown in Figure 1. The fraction of observers with μ below their upper limit is seen to vary with μ, just touching 84.13% for some rates, and greater than 84.13% for all others. The requirement that at least 84.13% of the observers have μ below their upper limit is therefore satisfied for all μ, and, hence, the upper limit definition is a reasonable one from this point of view.

The structure that is seen in Figure 1 is real and is not due to the finite computation step size of μ; each period of the variation contains about 5 points. The reason for the structure can be best understood by considering an example. The upper limit of 84.13% confidence for a measurement of 6 events in a unit time is 9.6. Therefore, if μ=9.5, all observers measuring 6, 7, 8, ... will have the average rate below their upper limit, and those measuring 0, 1, 2, 3, 4 or 5 events will not.

However, as μ increases past 9.6, suddenly the observers measuring 6 events will not have the rate below their upper limit any more, and the fraction of observers that do will suddenly decrease (to exactly 84.13%). Now, only observers that measure 7 or more events in a unit time will have the rate below their upper error bar. As μ increases, the fraction of observers with 7 or more events slowly increases and the plotted fraction therefore increases. When μ exceeds 10.8, which is the upper limit for 7 events, the fraction again decreases, and so on. Note that as μ gets larger the variations decrease and the fraction approaches a constant value of 84.13%. This is as expected since the discrete Poisson distribution becomes a continuous Gaussian distribution in the limit of large μ, and the requirement for a continuous distribution is that exactly ε% of the observers have the rate below their upper limit.

Figures 2a and b show the fraction of observers with μ above their lower limit for the standard and maximum likelihood lower limit respectively. Since the two lower limit definitions are not the same, the results are seen to be different. The standard lower limit plot (Figure 2a) is similar in character to that of the upper limit (Figure 1). The fraction of observers with μ above their lower limits is always greater than or equal to 84.13%. For the maximum likelihood case (Figure 2b), however, the fraction is seen to be less than or equal to 84.13%. Therefore the standard limits, but not the maximum likelihood lower limits, satisfy our requirement.

Summary

1) The standard upper and lower limits of 84.13% confidence satisfy the criterion that at least 84.13% of the observers measuring a fixed rate of events (with observations Poisson distributed about the rate) have the rate below (above) their upper (lower) limits.
2) The maximum likelihood upper limits are the same as the standard ones and therefore also satisfy the above criterion.
3) The maximum likelihood lower limits are different from the standard ones and do not satisfy the criterion given above.

Appendix

The Poisson likelihood function for rate μ given a measurement of n events in a unit time is

$$L(\mu, n) = \frac{e^{-\mu}\,\mu^{n}}{n!} \tag{A1}$$

(Orear, 1958).

Upper and lower limits of confidence ε are defined by

$$\frac{\int_0^{\mu_u} L(\mu,n)\,d\mu}{\int_0^{\infty} L(\mu,n)\,d\mu} = \varepsilon \tag{A2a}$$

$$\frac{\int_{\mu_l}^{\infty} L(\mu,n)\,d\mu}{\int_0^{\infty} L(\mu,n)\,d\mu} = \varepsilon \tag{A2b}$$

(Orear, 1958). From Gradshteyn and Ryzhik (1965, eqn 3.381 #4),

$$\int_0^{\infty} \frac{e^{-\mu}\,\mu^{n}}{n!}\,d\mu = \frac{\Gamma(n+1)}{n!} = 1\ . \tag{A3}$$

Therefore

$$\int_0^{\mu_u} \frac{e^{-\mu}\mu^{n}}{n!}\,d\mu = \varepsilon \tag{A4a}$$

$$\int_{\mu_l}^{\infty} \frac{e^{-\mu}\mu_{n}}{n!}\,d\mu = \varepsilon\ . \tag{A4b}$$

Using $\int_0^z x^{p-1} e^{-x}\, dx = e^{-z} \sum_{k=0}^{\infty} \frac{z^{p+k}}{p\,(p+1).......(p+k)}$ (Gradshteyn and

Ryzhik, 1965, eqn 3.381 #2) in addition to A3 and the fact that

$\sum_0^{\infty} \frac{e^{-\mu} \mu^n}{n!} = 1$, we find that

$$\sum_{k=0}^{n} \frac{e^{-\mu_u} \mu_u^k}{k!} = 1 - \varepsilon \qquad \text{(A5a)}$$

$$\sum_{k=n+1}^{\infty} \frac{e^{-\mu_l} \mu_l^k}{k!} = 1 - \varepsilon \,. \qquad \text{(A5b)}$$

References

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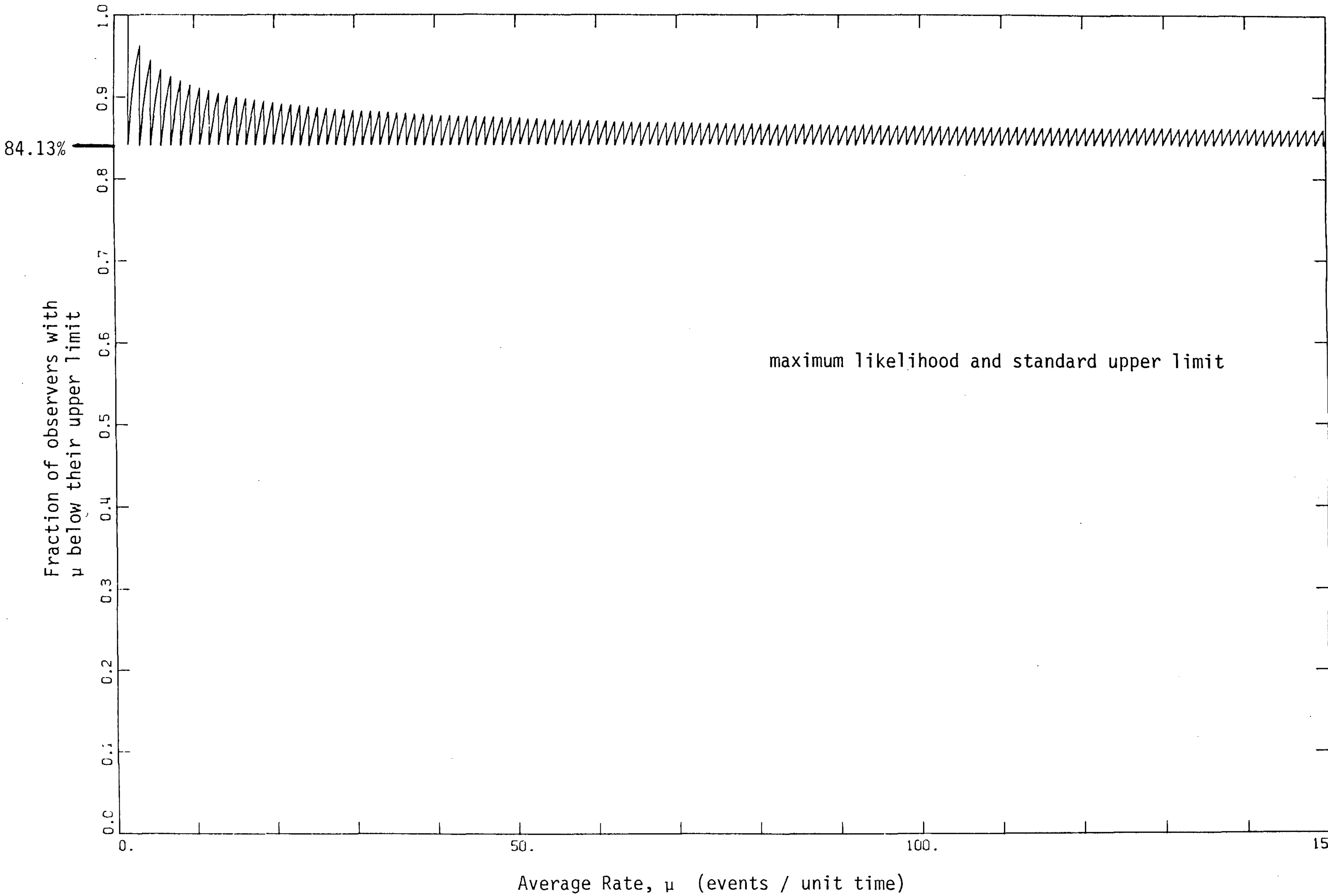

84.13%
maximum likelihood and standard upper limit
Fraction of observers with
μ below their upper limit
Average Rate, μ (events / unit time)
0.
50.
100.
0.0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure 1

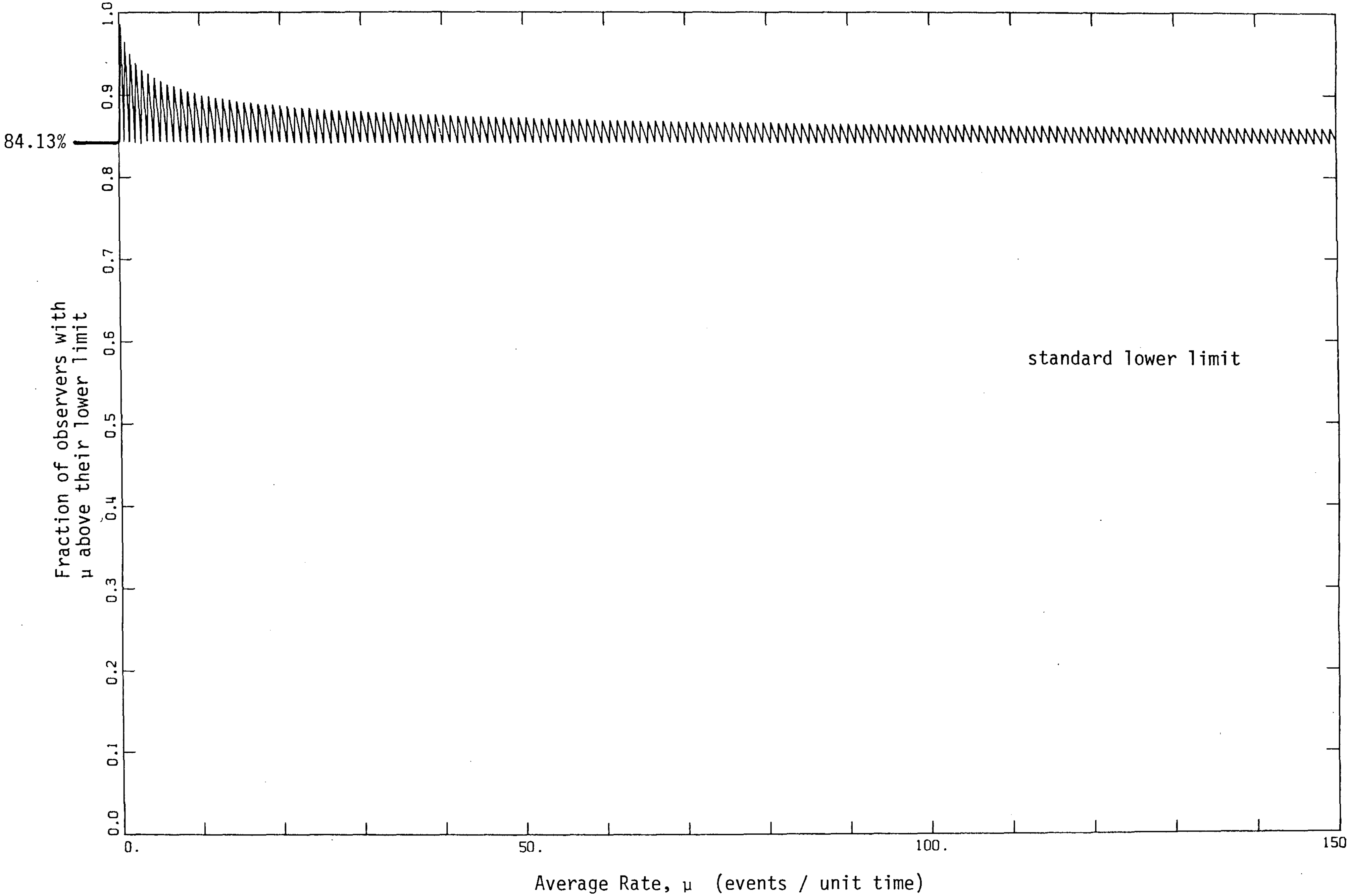
84.13%
Fraction of observers with
μ above their lower limit
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
standard lower limit
0.
50.
100.
150
Average Rate, μ (events / unit time)

Figure 2a

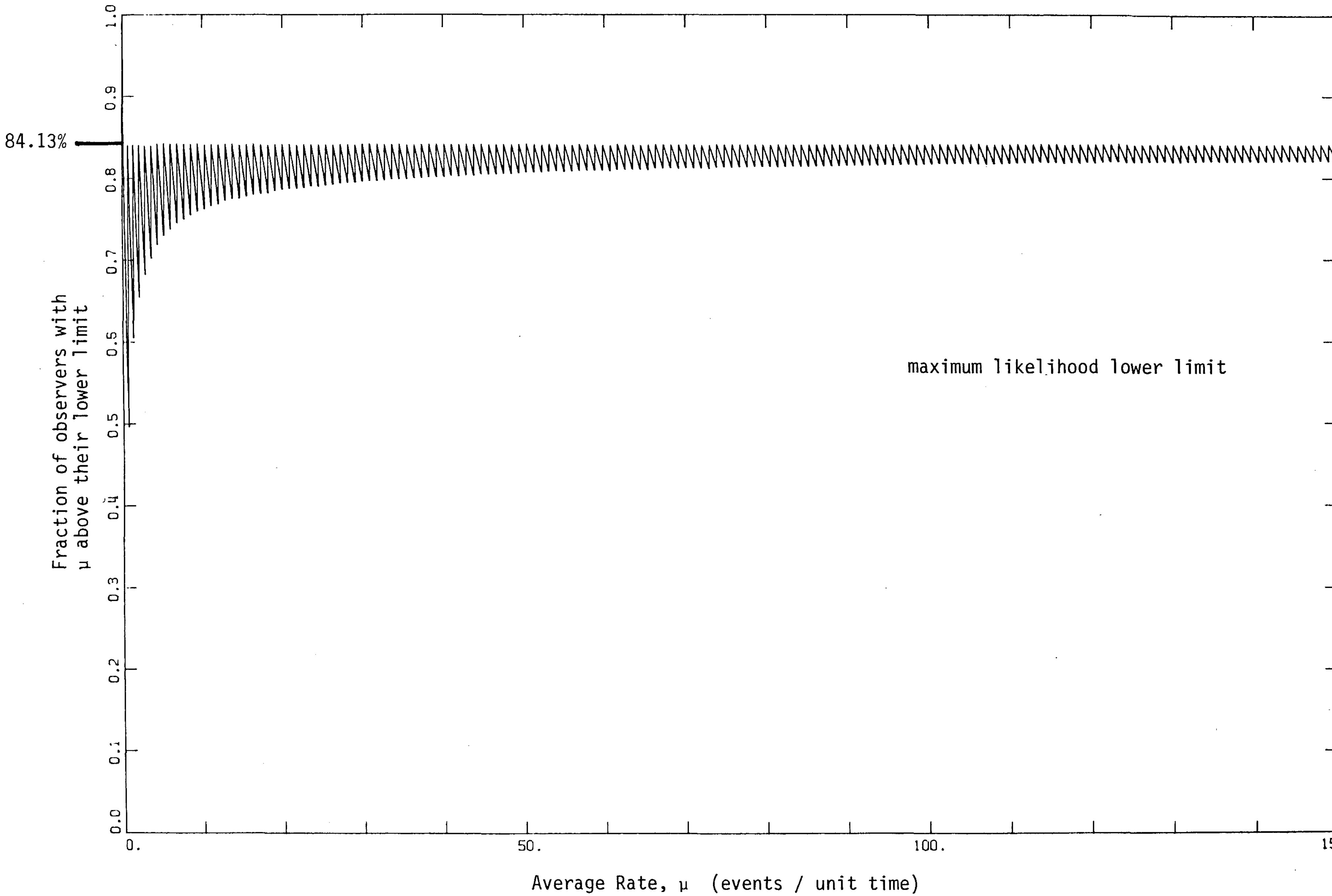
84.13%
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Fraction of observers with
μ above their lower limit
maximum likelihood lower limit
0.
50.
100.
150
Average Rate, μ (events / unit time)

Figure 2b