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A Technique for Determining Particle Anisotropies using the Voyager Low Energy Telescopes

by

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Abstract

In regions of space where there is no apriori information about the orientation of possible particle anisotropies, the four Voyager Low Energy Telescopes (LETs) can be used to specify the direction and magnitude of any first-order anisotropy. If directions in space can be identified that represent symmetry axes for particle motion (e.g. magnetic field lines or an observer's direction of motion with respect to a particle population), the LETs can be used to determine the magnitudes of higher-order, as well as first-order, anistotropies about these directions. The coefficients of the linear equations that relate the anisotropy magnitudes to the telescope counting rates are calculated in this report.

The counting rate of a particle telescope in a flux $F(\vartheta, \varphi, E)$ is

$$C = \int_{E_1}^{E_2} dE \int_{\Omega} d\Omega \, F(\vartheta, \varphi, E) \; A(\vartheta, \varphi) \qquad (1)$$

where $E_1 - E_2$ is the energy range of particles whose energy loss in the detectors exceeds the counting thresholds and $A(\vartheta, \varphi)$ is the effective area of the telescope (e.g. the overlap area of two detectors). For an isotropic flux, j(E),

$$C = \int_{E_1}^{E_2} dE \, j(E) \int_{\Omega} d\Omega \, A(\vartheta, \varphi) = A\Omega \int_{E_1}^{E_2} dE \, j(E)$$

where $A\Omega \equiv \int_{\Omega} d\Omega A(\vartheta, \varphi)$ is the telescope geometry factor. For an anisotropic flux, one would like to use the counting rates in two or more independent telescopes to obtain information about the angular dependence of $F(\vartheta, \varphi, E)$ or, assuming F is separable as $F(\vartheta, \varphi, E) = I(\vartheta, \varphi) j(E)$, about $I(\vartheta, \varphi)$ The specific application considered in this report is the four LETs that are part of the Cosmic Ray Subsystem on the Voyager spacecraft. The approach here will be to expand $I(\vartheta, \varphi)$ in a series of spherical harmonics and do the angular integration in equation 1 numerically for each term of the series using the appropriate formula for $A(\vartheta, \varphi)$. The resulting linear equations can then be solved to give the coefficients in the spherical harmonic expansion in terms of the four counting rates.

Since the spherical harmonics $Y_{lm}(\vartheta, \varphi)$ are a complete set of functions, any function $I(\vartheta, \varphi)$ (with sufficient continuity properties) can be expressed as a series in terms of them,

$$I(\vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \; Y_{lm}(\vartheta, \varphi)$$

where the a_{lm} are constants. The LETs give four independent equations that can be used to determine four of the constants. Given no additional information, we chose to determine the first four a_{lm}. The expansion is most convenient in terms of the spherical harmonics defined by Morse and Feshbach (1953, p.1271):

$$I(\vartheta, \varphi) = a_{00} + a_{10}\cos\vartheta + a_{11}\sin\vartheta\cos\varphi + a_{1-1}\sin\vartheta\sin\varphi .$$

This equation can be put into a simpler form by rotating the coordinate system by the Euler angles (Goldstein, 1950, p. 107) $\vartheta_{\varepsilon 1}$, $\varphi_{\varepsilon 1}$, and $\psi_{\varepsilon 1}$ such that

$$\cos\vartheta_{\varepsilon 1} = \frac{a_{10}}{\sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}} \quad \sin\vartheta_{\varepsilon 1} = \frac{\sqrt{a_{11}^2 + a_{1-1}^2}}{\sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}}$$

$$\cos\varphi_{\varepsilon 1} = -\frac{a_{1-1}}{\sqrt{a_{11}^2 + a_{1-1}^2}} \quad \sin\varphi_{\varepsilon 1} = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{1-1}^2}}$$

$$\psi_{\varepsilon 1} = \text{arbitrary} \qquad (\text{choose} = 0) .$$

The result is

$$I(\vartheta, \varphi) = a_0 + a_1\cos\vartheta_1 \tag{2}$$

where $a_0 = a_{00}$, $a_1 = \sqrt{a_{10}^2 + a_{11}^2 + a_{1-1}^2}$ and ϑ_1 is the angle relative to the new z-axis. This form shows explicitly that the LETs can measure the magnitude and direction of a Comptor-Getting-type anisotropy. The four parameters determined in this case are a_0, a_1, $\vartheta_{\varepsilon 1}$, and $\varphi_{\varepsilon 1}$.

In some environments, reasonable guesses can be made for the values of one or more of these constants and the LETs can then be used to specify the parameters of higher order spherical harmonics. For instance, in a magnetosphere one might assume that the only first-order anisotropy is due to the corotation of the particle distribution past the observer. In this case, the direction of the axis of the anisotropy ($\vartheta_{\varepsilon 1}$ and $\varphi_{\varepsilon 1}$) is known, and the extra two degrees of freedom can be used to determine $\cos^2\vartheta$ terms that pertain to pitch-angle distributions about the field lines.

If it is assumed that each higher order spherical harmonic term is axially symmetric about some axis, $\hat{e}_i$, it can be shown (Sanderson and

Hynds, 1977, and references therein) that $I(\vartheta, \varphi)$ can be represented as:

$$I(\vartheta, \varphi) = a_0 + a_1 \cos\vartheta_1 + a_2 \frac{3}{2}(\cos^2\vartheta_2 - \frac{1}{3}) + a_3 \frac{5}{2}(\cos^3\vartheta_3 - \frac{3}{5}\cos\vartheta_3) + \cdots \quad (3)$$

where ϑ_1, ϑ_2 and ϑ_3 are measured with respect to $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. There are clearly fewer parameters in the $\cos^2\vartheta$ and $\cos^3\vartheta$ terms in this equation than in the same order terms in the spherical harmonic equation. For the $\cos^2\vartheta$ term, there are five spherical harmonic coefficients a_{20}, a_{21}, a_{2-1}, a_{22} and a_{2-2}, but in equation 3 only three parameters appear; namely, a_2 and the Euler angles $\vartheta_{\varepsilon 2}$ and $\varphi_{\varepsilon 2}$ of the $\hat{e}_2$ axis. Similarly, for the a_3 term, the seven spherical harmonic coefficients correspond to the three parameters a_3, $\vartheta_{\varepsilon 3}$, and $\varphi_{\varepsilon 3}$ above. The reason is that, to be written in the form of equation 3, the a_{2m} and a_{3m} terms in the spherical harmonic expansions were assumed to be axially symmetric about some axes $\hat{e}_2$ and $\hat{e}_3$. The Maxwell theory of poles of the spherical harmonics (see e.g. Sanderson and Hynds, 1977), from which equation 3 was derived, requires, in general, two axes $\hat{e}_{2a}$ and $\hat{e}_{2b}$ to represent the a_{2m} spherical harmonic terms. There are therefore five parameters, a_2, $\vartheta_{\varepsilon 2a}$, $\varphi_{\varepsilon 2a}$, $\vartheta_{\varepsilon 2b}$ and $\varphi_{\varepsilon 2b}$ to match the five a_{2m}. For the a_{3m} terms, three axes (six Euler angles) are required. The axial symmetry assumption means that two a_2 axes line up and that the three a_3 axes line up. The expansion for $I(\vartheta, \varphi)$ given in equation 3 should therefore only be used if there is some physical basis for assuming axially symmetric $\cos^2\vartheta$ or $\cos^3\vartheta$ distributions. In the previous magnetosphere example, the field lines impose a natural symmetry condition on the pitch angle distributions.

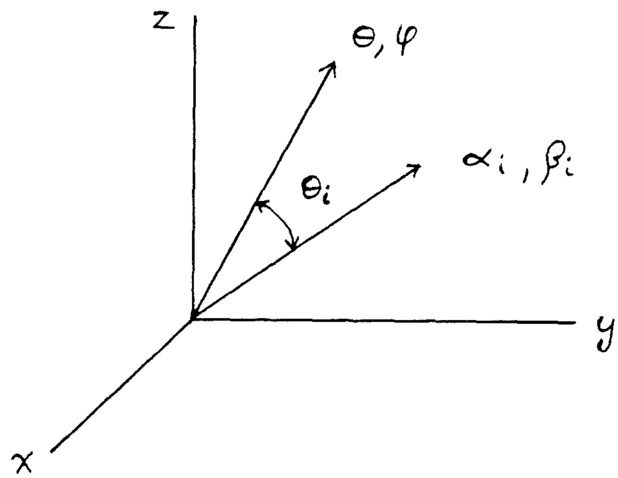
The next step is to perform the angular integration in equation 1. For $F(E, \vartheta, \varphi) = j(E)\, I(\vartheta, \varphi)$,

$$C = \int_{E_1}^{E_2} dE\, j(E) \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta d\vartheta\ A(\vartheta, \varphi)\, [a_0 + a_1 \cos\vartheta_1 + a_2 \frac{3}{2}(\cos^2\vartheta_2 - \frac{1}{3}) + a_3 \frac{5}{2}(\cos^3\vartheta_3 - \frac{3}{5}\cos\vartheta_3)]. \quad (4)$$

In order to actually do the integrals ϑ_i must be transformed into ϑ and φ using

$$\cos\vartheta_i = \cos\vartheta\cos\alpha_i + \sin\vartheta\sin\alpha_i\cos(\varphi-\beta_i) \qquad i = 1,2,3,... \qquad (5)$$

where α_i and β_i are the angles of the i^{th} symmetry axis ($\hat{e}_i$) in the telescope coordinate frame (z axis = telescope symmetry axis).



It is shown in the Appendix that, as one would expect from the cylindrical symmetry, the integral does not depend on β_i. We also need an explicit expression for $A(\vartheta, \varphi)$. For the LET case the area-overlap formula for two separated circular disks can be used:

$$\begin{aligned} A(\vartheta,\varphi) &= \pi r_s^2\cos\vartheta && 0\le\vartheta<\vartheta_c \\ &= \cos\vartheta\left\{\frac{r_1^2}{2}\left[2\Phi_1-\sin2\Phi_1\right]+\frac{r_2^2}{2}\left[2\Phi_2-\sin2\Phi_2\right]\right\} && \vartheta_c\le\vartheta<\vartheta_m \\ &= 0 && \vartheta_m\le\vartheta \end{aligned}$$

(Sullivan,1971) where:

r_1 = radius of detector 1

r_2 = radius of detector 2

x_s = smaller of r_1 and r_2

$$\vartheta_c = \tan^{-1}\frac{|r_1 - r_2|}{d}$$

$$\vartheta_m = \tan^{-1}\frac{(r_1 + r_2)}{d}$$

$$\Phi_1 = \cos^{-1}\left(\frac{r_1^2 + d^2\tan^2\vartheta - r_2^2}{2dr_1\tan\vartheta}\right)$$

$$\Phi_2 = \cos^{-1}\left(\frac{r_2^2 + d^2\tan^2\vartheta - r_1^2}{2dr_2\tan\vartheta}\right)$$

d = detector separation

Note that due to the assumed cylindrical symmetry, $A(\vartheta, \varphi)$ does not depend on φ.

The integrals can now be done. A convenient way to express the answer is to define

$$K_0 = 2\pi\int_0^\pi d\vartheta\, A(\vartheta)\sin\vartheta = A\Omega$$

$$K_1(\alpha_1) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta\, A(\vartheta)\cos\vartheta_1\sin\vartheta \tag{7}$$

$$K_2(\alpha_2) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta\, A(\vartheta)\frac{3}{2}\left(\cos^2\vartheta_2 - \frac{1}{3}\right)\sin\vartheta$$

$$K_3(\alpha_3) = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta\, A(\vartheta)\,\frac{5}{2}\left(\cos^3\vartheta_3 - \frac{3}{5}\cos\vartheta_3\right)\sin\vartheta$$

Comparing equations 4 and 7, we see that the counting rate in a telescope can now be succinctly expressed as

$$C = \int_{E_1}^{E_2} dE\, j(E) \sum_{i=0}^{n} a_i K_i(\alpha_i)\ . \tag{8}$$

In the case of the LETs there are four such equations so that in principle a_0, a_1, a_2, and a_3 can be determined. In the Appendix, the values of K_0, K_1, K_2, and K_3 are calculated for both a window-L1 and a L1-L2 configuration. The results are listed in the computer output at the end of this report. As is indicated in equation 8 and shown in the Appendix, the values of K_i depend on the angles α_i between the telescope axis and the i^{th} symmetry axis. Fortunately, however, the α_i dependence comes out of the integrals, so that the integrals only need be done once for a given detector configuration. For example, using the numbers listed in the computer output, the fully expanded expression for the counting rate for the nominal L1-L2 configuration (r_1=r_2=0.95 cm, d=4.08 cm) is

$$C = \int_{E_1}^{E_2} dE\, j(E)\, [a_0 \cdot 0.4367 + a_1 \cdot 0.4263 \cdot \cos\alpha_1 +$$
$$\alpha_2(0.6091 \cdot \cos^2\alpha_2 - 0.2030) + \alpha_3(0.9432 \cdot \cos^3\alpha_3 - 0.5659 \cdot \cos\alpha_3)]$$

Appendix

In this section the equations for K_i (equations 7) are put into forms that can be numerically integrated given the detector configuration. First, since $A(\vartheta)$ does not depend on φ, all integrals involving φ can be done analytically. A useful equation is

$$\int_0^{2\pi} d\varphi \cos^n(\varphi-\beta_i) = \int_0^{2\pi} d\varphi \cos^n\varphi = \begin{cases} 2\pi & n=0 \\ 0 & n \text{ odd} \\ \pi & n=2,4,6,\ldots \end{cases}$$

As expected from the cylindrical symmetry, the β_i dependence drops out so that β_i can be set to zero in equation 5.

K_0:

$$K_0 = 2\pi \int_0^{\pi} d\vartheta\, A(\vartheta)\sin\vartheta$$

$K_1(\alpha_1)$:

$$\cos\vartheta_1 = \cos\vartheta\cos\alpha_1 + \cos\varphi \text{ terms}$$

$$K_1(\alpha_1) = \cos\alpha_1 2\pi \int_0^{\pi} d\vartheta\, A(\vartheta)\cos\vartheta\sin\vartheta$$

$K_2(\alpha_2)$:

$$\cos^2\vartheta_2 = \cos^2\vartheta\cos^2\alpha_2 + \sin^2\vartheta\sin^2\alpha_2\cos^2\varphi + \cos\varphi \text{ terms}$$
$$= \cos^2\alpha_2\left(\cos^2\vartheta - \sin^2\vartheta\cos^2\varphi\right) + \sin^2\vartheta\cos^2\varphi + \cos\varphi \text{ terms}$$

$$K_2(\alpha_2) = \frac{3}{2}\cos^2\alpha_2\left[2\pi\int_0^{\pi} d\vartheta\, A(\vartheta)\cos^2\vartheta\sin\vartheta - \int_0^{2\pi} d\varphi\cos^2\varphi \int_0^{2\pi} d\vartheta A(\vartheta)\sin^2\vartheta\sin\vartheta\right]$$
$$+ \frac{3}{2}\int_0^{2\pi} d\varphi\cos^2\varphi \int_0^{2\pi} d\vartheta\, A(\vartheta)\sin^2\vartheta\sin\vartheta - \frac{1}{2}2\pi\int_0^{\pi} d\vartheta A(\vartheta)\sin\vartheta$$

$$= \frac{3\pi}{2}\cos^2\alpha_2\left[2\int_0^\pi d\vartheta\,A(\vartheta)\cos^2\vartheta\sin\vartheta - \int_0^\pi d\vartheta\,A(\vartheta)\sin^3\vartheta\right]$$

$$+\ \pi\left[\frac{3}{2}\int_0^\pi d\vartheta\,A(\vartheta)\sin^3\vartheta - \int_0^\pi d\vartheta\,A(\vartheta)\sin\vartheta\right]$$

$K_3(\alpha_3)$:

$$\cos^3\vartheta_3 = \cos^3\vartheta\cos^3\alpha_3 + 3\cos\vartheta\cos\alpha_3\sin^2\vartheta\sin^2\alpha_3\cos^2\varphi + \text{odd powers of }\cos\varphi$$

$$= \cos^3\alpha_3(\cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta\cos^2\varphi)\ +$$

$$\cos\alpha_3 3\cos\vartheta\sin^2\vartheta\cos^2\varphi\ +\ \text{odd powers of }\cos\varphi$$

$$K_3(\alpha_3) = \frac{5}{2}\cos^3\alpha_3\left[2\pi\int_0^\pi d\vartheta\,A(\vartheta)\cos^3\vartheta\sin\vartheta - 3\int_0^{2\pi} d\varphi\cos^2\varphi\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin^3\vartheta\right]$$

$$+\ \cos\alpha_3\left[\frac{3\cdot 5}{2}\int_0^{2\pi} d\varphi\cos^2\varphi\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin^3\vartheta - 2\pi\frac{3}{2}\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin\vartheta\right]$$

$$= \frac{5}{2}\pi\cos^3\alpha_3\left[2\int_0^\pi d\vartheta\,A(\vartheta)\cos^3\vartheta\sin\vartheta - 3\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin^3\vartheta\right]$$

$$-\ \pi\cos\alpha_3\left[\frac{15}{2}\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin^3\vartheta - 3\int_0^\pi d\vartheta\,A(\vartheta)\cos\vartheta\sin\vartheta\right]$$

The program **ANNIE** performs these integrals numerically for given detector configurations defined by r_1, r_2 and d (equation 6). An output and a program listing are at the end of this report. Note that K_0 is the geometry factor of the telescope and can be analytically calculated (Sullivan, 1968). The values of K_0 returned by the program are consistent with such calculations. The test case listed on the fourth page of the output is for two detectors separated by a large distance compared to their radii, so that $A(\vartheta)$ becomes essentially a delta function centered at

$\vartheta=0$. Considering the above equations, one can show that the program gives the expected result:

$$C = \int_{E_1}^{E_2} dE\, j(E)\cdot A\Omega\cdot[a_0 + a_1\cos\alpha_1 +$$

$$a_2\frac{3}{2}(\cos^2\alpha_2-\frac{1}{3}) + a_3\frac{5}{2}(\cos^3\alpha_3-\frac{3}{5}\cos\alpha_3)]$$

where $A\Omega=K_0=9.87\text{x}10^{-6}$. Note the similarity in form between this equation and equation 3.

References

Goldstein, H., *Classical Mechanics*, Addison-Wesley, Massachusetts, 1950.

Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, part II, McGraw-Hill, New York, 1953.

Sanderson, T. R. and R. J. Hynds, Multiple telescope measurements of particle anisotropies in space, *Planet. Space Sci.*, *25*, 799, 1977.

Sullivan, J. D., Geometrical factor and directional response of single and multi-element particle telescopes, *Nuclear Instr. and Methods*, *95*, 5, 1971.

```
                         NOMINAL L1-L2
R1 = 0.9500      R2 = 0.9500      D =  4.080
THETAC =  0.0     THETAM =  0.436      DELTH = 0.0020

INTEGRAL(A SIN)=0.06950           INTEGRAL(A COS SIN)=0.06785
INTEGRAL(A COS**2 SIN)=0.06626    INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06473    INTEGRAL(A COS SIN**3)=0.00312


 K0 = A OMEGA =   0.4367
 K1 =    COS(ALPHA1) *  0.4263
 K2 =    COS**2(ALPHA2) *   0.6091    +  -0.2030
 K3 =    COS**3(ALPHA3) *   0.9432    +    COS(ALPHA3) * -0.5659
```

```
                         NOMINAL WINDOW-L1
R1 = 1.7400      R2 = 0.9500      D =  1.550
THETAC =  0.471   THETAM =  1.048      DELTH = 0.0020

INTEGRAL(A SIN)=0.73153           INTEGRAL(A COS SIN)=0.61663
INTEGRAL(A COS**2 SIN)=0.52847    INTEGRAL(A SIN**3)=0.20308
INTEGRAL(A COS**3 SIN)=0.45957    INTEGRAL(A COS SIN**3)=0.15709


 K0 = A OMEGA =   4.5964
 K1 =    COS(ALPHA1) *  3.8744
 K2 =    COS**2(ALPHA2) *   4.0238    +  -1.3412
 K3 =    COS**3(ALPHA3) *   3.5175    +    COS(ALPHA3) * -2.1102
```

```
                         VGR 1  LET A     L1-L2
R1 = 0.9505      R2 = 0.9484      D =  4.076
THETAC =  0.001   THETAM =  0.436      DELTH = 0.0020

INTEGRAL(A SIN)=0.06947           INTEGRAL(A COS SIN)=0.06782
INTEGRAL(A COS**2 SIN)=0.06623    INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06470    INTEGRAL(A COS SIN**3)=0.00312


 K0 = A OMEGA =   0.4365
 K1 =    COS(ALPHA1) *  0.4261
 K2 =    COS**2(ALPHA2) *   0.6088    +  -0.2029
 K3 =    COS**3(ALPHA3) *   0.9427    +    COS(ALPHA3) * -0.5656
```

```
                    VGR 1   LET B     L1-L2
R1 = 0.9479      R2 = 0.9519      D =   4.081
THETAC =   0.001    THETAM =   0.436      DELTH = 0.0020

INTEGRAL(A SIN)=0.06945             INTEGRAL(A COS SIN)=0.06779
INTEGRAL(A COS**2 SIN)=0.06620      INTEGRAL(A SIN**3)=0.00324
INTEGRAL(A COS**3 SIN)=0.06467      INTEGRAL(A COS SIN**3)=0.00312


 K0 = A OMEGA =   0.4363
 K1 =    COS(ALPHA1) *   0.4259
 K2 =    COS**2(ALPHA2) *   0.6086    +  -0.2029
 K3 =    COS**3(ALPHA3) *   0.9425    +     COS(ALPHA3) * -0.5655
```

```
                    VGR 1   LET C     L1-L2
R1 = 0.9487      R2 = 0.9478      D =   4.079
THETAC =   0.000    THETAM =   0.435      DELTH = 0.0020

INTEGRAL(A SIN)=0.06905             INTEGRAL(A COS SIN)=0.06741
INTEGRAL(A COS**2 SIN)=0.06583      INTEGRAL(A SIN**3)=0.00322
INTEGRAL(A COS**3 SIN)=0.06431      INTEGRAL(A COS SIN**3)=0.00309


 K0 = A OMEGA =   0.4338
 K1 =    COS(ALPHA1) *   0.4235
 K2 =    COS**2(ALPHA2) *   0.6053    +  -0.2018
 K3 =    COS**3(ALPHA3) *   0.9374    +     COS(ALPHA3) * -0.5624
```

```
                    VGR 1   LET D     L1-L2
R1 = 0.9460      R2 = 0.9466      D =   4.075
THETAC =   0.000    THETAM =   0.435      DELTH = 0.0020

INTEGRAL(A SIN)=0.06863             INTEGRAL(A COS SIN)=0.06700
INTEGRAL(A COS**2 SIN)=0.06544      INTEGRAL(A SIN**3)=0.00319
INTEGRAL(A COS**3 SIN)=0.06393      INTEGRAL(A COS SIN**3)=0.00307


 K0 = A OMEGA =   0.4312
 K1 =    COS(ALPHA1) *   0.4210
 K2 =    COS**2(ALPHA2) *   0.6017    +  -0.2006
 K3 =    COS**3(ALPHA3) *   0.9320    +     COS(ALPHA3) * -0.5592
```

```
                     VGR 2  LET A     L1-L2
R1 = 0.9471      R2 = 0.9491       D =  4.075
THETAC =  0.000    THETAM =  0.436       DELTH = 0.0020

INTEGRAL(A SIN)=0.06913             INTEGRAL(A COS SIN)=0.06748
INTEGRAL(A COS**2 SIN)=0.06590      INTEGRAL(A SIN**3)=0.00323
INTEGRAL(A COS**3 SIN)=0.06438      INTEGRAL(A COS SIN**3)=0.00310


K0 = A OMEGA =   0.4344
K1 =    COS(ALPHA1) *  0.4240
K2 =    COS**2(ALPHA2) *  0.6059   +  -0.2020
K3 =    COS**3(ALPHA3) *  0.9383   +    COS(ALPHA3) * -0.5630


                     VGR 2  LET B     L1-L2
R1 = 0.9451      R2 = 0.9483       D =  4.064
THETAC =  0.001    THETAM =  0.436       DELTH = 0.0020

INTEGRAL(A SIN)=0.06908             INTEGRAL(A COS SIN)=0.06743
INTEGRAL(A COS**2 SIN)=0.06585      INTEGRAL(A SIN**3)=0.00323
INTEGRAL(A COS**3 SIN)=0.06433      INTEGRAL(A COS SIN**3)=0.00310


K0 = A OMEGA =   0.4340
K1 =    COS(ALPHA1) *  0.4237
K2 =    COS**2(ALPHA2) *  0.6054   +  -0.2018
K3 =    COS**3(ALPHA3) *  0.9373   +    COS(ALPHA3) * -0.5624


                     VGR 2  LET C     L1-L2
R1 = 0.9502      R2 = 0.9424       D =  4.084
THETAC =  0.002    THETAM =  0.434       DELTH = 0.0020

INTEGRAL(A SIN)=0.06835             INTEGRAL(A COS SIN)=0.06674
INTEGRAL(A COS**2 SIN)=0.06519      INTEGRAL(A SIN**3)=0.00317
INTEGRAL(A COS**3 SIN)=0.06369      INTEGRAL(A COS SIN**3)=0.00304


K0 = A OMEGA =   0.4295
K1 =    COS(ALPHA1) *  0.4193
K2 =    COS**2(ALPHA2) *  0.5994   +  -0.1998
K3 =    COS**3(ALPHA3) *  0.9287   +    COS(ALPHA3) * -0.5572
```

```
                   VGR 2  LET D    L1-L2
R1 = 0.9499      R2 = 0.9447      D =  4.061
THETAC =  0.001   THETAM =  0.437      DELTH = 0.0020

INTEGRAL(A SIN)=0.06934           INTEGRAL(A COS SIN)=0.06768
INTEGRAL(A COS**2 SIN)=0.06609    INTEGRAL(A SIN**3)=0.00325
INTEGRAL(A COS**3 SIN)=0.06456    INTEGRAL(A COS SIN**3)=0.00312


K0 = A OMEGA =  0.4357
K1 =   COS(ALPHA1) *  0.4252
K2 =   COS**2(ALPHA2) *  0.6075   +  -0.2025
K3 =   COS**3(ALPHA3) *  0.9405   +    COS(ALPHA3) * -0.5643
```

```
                   TEST CASE
R1 =  1.CCOE 00      R2 =  1.000E 00      D =  1.000E 03
THETAC =  0.0         THETAM =  2.000E-03       DELTH =  1.000E-05

INTEGRAL(A SIN)= 1.571E-06           INTEGRAL(A CCS SIN)= 1.571E-06
INTEGRAL(A COS**2 SIN)= 1.571E-06    INTEGRAL(A SIN**3)= 1.571E-12
INTEGRAL(A COS**3 SIN)= 1.571E-06    INTEGRAL(A COS SIN**3)= 1.571E-1

K0 = A OMEGA = 9.869E-06
K1 =   CCS(ALPHA1) * 9.869E-06
K2 =   CCS**2(ALPHA2) * 1.480E-05   + -4.935E-06
K3 =   CCS**3(ALPHA3) * 2.467E-05   +    CCS(ALPHA3) *-1.480E-05
```

```
//ANNIE     JOB      (98221,CZG,SRL),'N. GEHRELS',TIME=(2,00)
//       EXEC FORTG
//FORT    DD    *
C -----
C            READ IN DETECTOR RADII AND SEPARATION
C -----
      DIMENSION LABEL(19)
      COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
    5 READ(5,1111,END=999) LABEL
1111  FORMAT(19A4)
      READ(5,1112) R1,R2,D
1112  FORMAT(3F10.0)
C -----
C            INITIALIZE VARIABLES
C -----
      CALL INIT(R1,R2,D)
      DELTH=0.002
C -----
C            OUTPUT THE A HEADER
C -----
      WRITE(6,3333) LABEL
3333  FORMAT(////1X19A4)
      WRITE(6,3334) R1,R2,D,THETAC,THETAM,DELTH
3334  FORMAT(' R1 = ',F6.4,'     R2 = ',F6.4,
     +  '     D = ',F6.3,/' THETAC = ',F6.3,
     +  '   THETAM = ',F6.3,'      DELTH = ',F6.4)
C -----
C            ZERO INTEGRATION SUMS AND INTEGRATE TERMS
C            INVOLVING A(THETA)*POWERS OF COS(THETA) AND SIN(THETA)
C            FROM THETA=0 TO THETA=THETAM
C -----
      SMS=0.
      SMCS=0.
      SMC2S=0.
      SMS3=0.
      SMC3S=0.
      SMCS3=0.
      THETA=0.
   10 THETA=THETA+DELTH
      IF(THETA.GT.THETAM) GOTO 20
      ATH=A(THETA)
      SINTH=SIN(THETA)
      COSTH=COS(THETA)
      SMS=SMS+ATH*SINTH
      SMCS=SMCS+ATH*COSTH*SINTH
      SMC2S=SMC2S+ATH*COSTH**2*SINTH
      SMS3=SMS3+ATH*SINTH**3
      SMC3S=SMC3S+ATH*COSTH**3*SINTH
      SMCS3=SMCS3+ATH*COSTH*SINTH**3
      GOTO 10
```

```
C -----
C          SCALE BY DELTH
C -----
 20   SMS=SMS*DELTH
      SMCS=SMCS*DELTH
      SMC2S=SMC2S*DELTH
      SMS3=SMS3*DELTH
      SMC3S=SMC3S*DELTH
      SMCS3=SMCS3*DELTH
      WRITE(6,4444) SMS,SMCS,SMC2S,SMS3,SMC3S,SMCS3
4444  FORMAT(/' INTEGRAL(A SIN)=',F7.5,10X'INTEGRAL(A COS SIN)=',
     +  F7.5/' INTEGRAL(A COS**2 SIN)=',F7.5,
     +  '    INTEGRAL(A SIN**3)=',F7.5/' INTEGRAL(A COS**3 SIN)=',
     +  F7.5,'    INTEGRAL(A COS SIN**3)=',F7.5)
C -----
C          CALCULATE THE VARIOUS TERMS
C -----
      TRMK01=2.*PI*SMS
      TRMK11=2.*PI*SMCS
      TRMK21=(3.*PI/2.)*(2.*SMC2S-SMS3)
      TRMK22=PI*(3.*SMS3/2.-SMS)
      TRMK31=5.*PI*(2.*SMC3S-3.*SMCS3)/2.
      TRMK32=PI*(15.*SMCS3/2.-3.*SMCS)
C -----
C          OUTPUT THE TERMS
C -----
      WRITE(6,5555) TRMK01,TRMK11,TRMK21,TRMK22,TRMK31,TRMK32
5555  FORMAT(////'  K0 = A OMEGA =',F8.4/'  K1 =    COS(ALPHA1) *',
     +  F8.4/'  K2 =    COS**2(ALPHA2) *',F8.4,'    + ',F8.4/
     +  '  K3 =    COS**3(ALPHA3) *',F8.4,'    +    COS(ALPHA3) *',
     +  F8.4///)
      GOTO 5
 999  STOP
      END
      SUBROUTINE INIT(R1,R2,D)
      COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
C -----
C          THIS SUBROUTINE INITIALIZES VARIABLE THAT ARE NEEDED
C          TO CALCULATED A(THETA)
C -----
      PI=3.14159
      RS=AMIN1(R1,R2)
      ABSR12=ABS(R1-R2)
      THETAC=ATAN(ABSR12/D)
      THETAM=ATAN((R1+R2)/D)
      R1SQR=R1*R1
      R2SQR=R2*R2
      RSSQR=RS*RS
      DSQR=D*D
      DR12=2*D*R1
      DR22=2*D*R2
      RETURN
      END
      FUNCTION A(THETA)
      COMMON PI,RS,THETAC,THETAM,R1SQR,R2SQR,RSSQR,DSQR,DR12,DR22
C -----
C          THIS FUNCTION CALCULATES THE PROJECTED OVERLAP AREA
C          OF THE TWO DETECTORS AS A FUNCTION OF THETA
```

```
C -----
      IF(THETA.GE.THETAC) GOTO 10
      A=PI*RSSQR*COS(THETA)
      RETURN
   10 IF(THETA.LT.THETAM) GOTO 20
      A=0.
      RETURN
   20 TANTH=TAN(THETA)
      PHI12=2.*ARCOS((R1SQR+DSQR*TANTH**2-R2SQR)/(DR12*TANTH))
      PHI22=2.*ARCOS((R2SQR+DSQR*TANTH**2-R1SQR)/(DR22*TANTH))
      A=COS(THETA)*(R1SQR*(PHI12-SIN(PHI12))/2.+
     +   R2SQR*(PHI22-SIN(PHI22))/2.)
      RETURN
      END
//DATA     DD  *
                             NOMINAL L1-L2
0.95        0.95        4.08
                             NOMINAL WINDOW-L1
1.74        0.95        1.55
                             VGR 1  LET A    L1-L2
0.9505      0.9484      4.076
                             VGR 1  LET B    L1-L2
0.9479      0.9519      4.081
                             VGR 1  LET C    L1-L2
0.9487      0.9478      4.079
                             VGR 1  LET D    L1-L2
0.9460      0.9466      4.075
                             VGR 2  LET A    L1-L2
0.9471      0.9491      4.075
                             VGR 2  LET B    L1-L2
0.9451      0.9483      4.064
                             VGR 2  LET C    L1-L2
0.9502      0.9424      4.084
                             VGR 2  LET D    L1-L2
0.9499      0.9447      4.061
//
$
```